A MATHEMATICAL VERIFICATION OF TRAJECTORY METHODS

By Lester Machta

U. S. Weather Bureau

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I. Introduction

Since upper air maps are prepared every six or twelve hours, it is necessary when computing trajectories from the wind field to make an estimate of the field of motion at intermediate times. Several techniques for accomplishing this interpolation of the field of motion in time are available. It was desired to study the errors which might be introduced by two of the simplest techniques. The most frequently-used trajectory interpolation procedures are identified by the phrases "Mid-point of successive streamline" and "Central tendency" methods. The former is illustrated graphically in Figure 1 and the latter in Figure 2 for the case of 12-hourly upper air maps.

To determine which of the methods is a superior technique, very simple mathematical models of airflow patterns were established from which the true trajectory could be computed. Further, by employing the mathematical definitions of the two techniques, the end points of the trajectories at given times were found. A comparison of these end points with the corresponding end point determined from the true trajectory yields the error which the approximate interpolation technique produces.

II. Constant Linear Wind Fields

A. Rotation - A field of pure rotation is described by the equations:

$$u = dx/dt = -by,$$
$$v = dy/dt = b\ (x-ct). \qquad (1)$$

Where b is the coefficient of rotation, c is the translatory velocity of the axes of rotation in the x-direction and t is time.

1. <u>True trajectory</u> - A solution of (1) which gives the true trajectory of a parcel is:

$$x = -(y_o + c/b)\sin bt + x_o \cos bt + ct,$$
$$y = x_o \sin bt + (y_o + c/b)\cos bt - c/b, \quad (2)$$

   where x_o, y_o, are the coordinates of the parcel at time t = 0. Equations (2) give the true coordinates along the trajectory at any time.

2. <u>Mid-point of successive streamlines method</u> - In the computation of trajectories by the mid-point of successive streamlines method, it is assumed that:

$$x_m = (x_1 + x_2)/2,$$
$$y_m = (y_1 + y_2)/2, \quad (3)$$

   where x_m, y_m, are the computed coordinates of the trajectory at the time of the second map; x_1, y_1 the displacement as measured from the first map only; and x_2, y_2 the displacement as measured from the second map. Since on any one map the systems are stationary, from the first map:

$$x_1 = -y_o \sin bT + x_o \cos bT,$$
$$y_1 = x_o \sin bT + y_o \cos bT, \qquad (4)$$

and from the second map:

$$x_2 = -y_o \sin bT + (x_o - cT) \cos bT + cT,$$
$$y_2 = (x_o - cT) \sin bT + y_o \cos bT, \qquad (5)$$

where T is the time interval between maps. The coordinates of a point along the trajectory at the time of the second map are, according to the mid-point of successive streamlines method:

$$x_m = -y_o \sin bT + (x_o - cT/2) \cos bT + cT/2,$$
$$y_m = (x_o - cT/2) \sin bT + y_o \cos bT. \qquad (6)$$

As can be seen from a comparison of equations (2) and (6), the deviation of the computed from the true position is independent of the initial coordinates x_o and y_o, since:

$$x - x_m = -(c/b) \sin bT + (cT/2) (\cos bT + 1),$$
$$y - y_m = (cT/2) \sin bT + (c/b) (\cos bT - 1). \qquad (7)$$

3. <u>Central tendency method</u> - In computing trajectories by the central tendency method, the

particle is followed on the first map for a time interval of T/2 and on the second map for the remainder of the interval between maps.

$$x_1' = -y_o \sin bT/2 + x_o \cos bT/2,$$
$$y_1' = x_o \sin bT/2 + y_o \cos bT/2, \qquad (8)$$

where x_1', y_1' are the coordinates at time T/2 after the time of the first map. The coordinates of the parcel, x_c, y_c, at the time of the second map, as computed by this method, are:

$$x_c = -y_1' \sin bT/2 + (x_1' - cT) \cos bT/2 + cT$$
$$= -y_o \sin bT + x_o \cos bT + cT (1 - \cos bT/2), \qquad (9)$$
$$y_c = (x_1' - cT) \sin bT/2 + y_1' \cos bT/2,$$
$$= x_o \sin bT + y_o \cos bT - cT \sin bT/2.$$

Again, the deviation of the computed trajectory from the true path is not a function of x_o, y_o:

$$x - x_c = (-c/b) \sin bT + cT \cos bT/2,$$
$$y - y_c = (c/b) (\cos bT - 1) + cT \sin bT/2. \qquad (10)$$

B. Deformation - A field of pure deformation is described by the equations:

$$dx/dt = ey,$$
$$dy/dt = e (x - ct), \qquad (11)$$

where e is the coefficient of deformation and the other notation is identical to that of the preceding section. A solution of (11) is:

$$x = (y_o - c/e) \sinh et + x_o \cosh et + ct,$$
$$y = x_o \sinh et + (y_o - c/e) \cosh et + c/e. \qquad (12)$$

With a development exactly analogous to that of the preceding section, the expressions obtained for the coordinates of a trajectory computed by the mid-point of successive streamlines method are:

$$x_m = y_o \sinh eT + (x_o - cT/2) \cosh eT + cT/2,$$
$$y_m = (x_o - cT/2) \sinh eT + y_o \cosh eT, \qquad (13)$$

and the deviations are:

$$x - x_m = -c/e \sinh eT + (cT/2)(1 + \cosh eT),$$
$$y - y_m = -c/e (\cosh eT - 1) + (cT/2) \sinh eT. \qquad (14)$$

Similarly, for the central tendency method:

$$x_c = y_o \sinh eT + x_o \cosh eT + cT (1 - \cosh eT/2),$$
$$y_c = x_o \sinh eT + y_o \cosh eT - cT \sinh eT/2, \qquad (15)$$

and

$$x - x_c = -(c/e) \sinh eT + cT \cosh eT/2,$$
$$y - y_c = cT \sinh eT/2 - (c/e)(\cosh eT - 1) \qquad (16)$$

Again, it is seen that for both methods the deviations from the true trajectory are independent of the initial coordinates.

C. Divergence - A field of pure divergence is described by the equations:

$$dx/dt = a(x - ct), \quad dy/dt = a\,y, \tag{17}$$

where a is the coefficient of divergence and the other notation is identical to that of the preceding sections. A solution of (17) is:

$$x = (x_o - c/a)\ e^{at} + ct + c/a, \quad y = y_o\ e^{at} \tag{18}$$

where e = 2.71828 (Base of natural logarithms).

Again, with a development analogous to that of section 2, the expressions for the coordinates of a trajectory computed by the mid-point of successive streamlines method are:

$$x_m = x_o\ e^{aT} + (cT/2)\ \ (1 - e^{aT}), \quad y_m = y_o e^{aT}. \tag{19}$$

and the deviations:

$$x - x_m = (cT/2)\ (1 + e^{aT}) + (c/a)\ (1 - e^{aT}), \quad y - y_m = 0. \tag{20}$$

Similarly, for the central tendency method:

$$x_c = x_o\ e^{aT} + cT\ (1 - e^{aT/2}), \quad y_c = y_o\ e^{aT}, \tag{21}$$

and

$$x - x_c = (c/a)\ (1 - e^{aT}) + cT\ e^{aT/2},$$
$$y - y_c = 0. \qquad (22)$$

Also, in this case, the deviations are independent of the initial position.

III. Varying Linear Wind Fields

In addition to constant fields of rotation, deformation, and divergence, it is also possible to treat cases in which such fields vary linearly with time. For simplicity, it is assumed that c is zero (i.e., that the axes of the field are stationary) in all cases investigated in this section.

A. Rotation - If we assume that the coefficient of rotation varies linearly with time from zero to a value of b in time t, then by definition:

$$b' = b/t. \qquad (23)$$

Substituting in (1) and letting $c = 0$:

$$dx/dt = -b'\ ty,$$
$$dy/dt = b'\ tx. \qquad (24)$$

Solution of (24) yields for the true trajectory:

$$x = -y_o \sin b't^2/2 + x_o \cos b't^2/2,$$
$$y = x_o \sin b't^2/2 + y_o \cos b't^2/2. \qquad (25)$$

If T is the time interval between maps and the coefficient of rotation varies linearly from zero to a value of b in this interval, the coordinates of the true trajectory at the time of the second map are, from (23) and (25):

$$x = -y_o \sin bT/2 + x_o \cos bT/2,$$
$$y = x_o \sin bT/2 + y_o \cos bT/2. \qquad (26)$$

From equations (26), the trajectory as determined from the mid-point of successive streamlines method can be found in a manner analogous to that of the preceding sections:

$$x_m = -(y_o/2) \sin bT + (x_o/2)(1 + \cos bT),$$
$$y_m = (x_o/2) \sin bT + (y_o/2)(1 + \cos bT). \qquad (27)$$

Similarly, for the central tendency method:

$$x_c = -y_o \sin bT/2 + x_o \cos bT/2,$$
$$y_c = x_o \sin bT/2 + y_o \cos bT/2. \qquad (28)$$

B. Deformation - Proceeding as in the case of rotation, for a linearly varying deformation field, let:

$$e' = e/t \qquad (29)$$

Then, if c is zero, from (11):

$$dx/dt = e'ty,$$
$$dy/dt = e'tx, \qquad (30)$$

which yields for the true position at the time of the second map:

$$x = y_o \sinh eT/2 + x_o \cosh eT/2, \qquad (31)$$
$$y = x_o \sinh eT/2 + y_o \cosh eT/2.$$

Applying the mid-point of successive streamlines method:

$$x_m = (y_o/2) \sinh eT + (x_o/2) (1 + \cosh eT), \qquad (32)$$
$$y_m = (x_o/2) \sinh eT + (y_o/2) (1 + \cosh eT).$$

and for the central tendency method:

$$x_c = y_o \sinh eT/2 + x_o \cosh eT/2, \qquad (33)$$
$$y_c = x_o \sinh eT/2 + y_o \cosh eT/2.$$

C. Divergence - Similarly, for the case of a linearly varying field of divergence, let:

$$a' = a/t. \qquad (34)$$

Then, if c is zero, from (17):

$$dx/dt = a'tx, \qquad (35)$$
$$dy/dt = a'ty,$$

which yields for the true position at the time of the second map:

$$x = x_o \, e^{aT/2}, \qquad (36)$$
$$y = y_o e^{aT/2}.$$

Applying the mid-point of successive streamlines method:

$$x_m = (x_o/2)\ (1 + e^{aT}),$$
$$y_m = (y_o/2)\ (1 + e^{aT}), \tag{37}$$

and for the central tendency method:

$$x_c = x_o\ e^{aT/2},$$
$$y_c = y_o\ e^{aT/2}, \tag{38}$$

It should be noted that in all the above cases involving linearly varying fields, the deviations of the computed trajectory from the true value are dependent upon the initial coordinates.

IV. Sinusoidal Wind Fields

A more realistic, yet simple, mathematical formulation of flow patterns in the atmosphere, especially at upper levels, is obtained by assuming a sinusoidal variation in the north-south component of the wind field superimposed on a constant west-east zonal current. If the axes are oriented so that the positive x-axis points toward the east and the positive y-axis towards the north, then:

$$u = dx/dt = U,$$
$$v = dy/dt = A(t)\ \sin 2\pi(x - ct)/L, \tag{39}$$

where:

U = constant zonal speed,

A (t) = amplitude factor (maximum value of the north-south component of the wind),

L = wavelength,

c = speed of the wave in the x-direction,

t = time.

Equations (39) can be solved with a variety of functions for $A(t)$.

A. Constant amplitude - For a wave moving in the x-direction with no changes in shape, the amplitude factor $A(t)$ is equal to a constant, A_o. For this case, a solution of (39) yields for the true trajectory:

$$x = Ut + x_o,$$
$$y = A_oL/2\pi c' \left[\cos 2\pi x_o/L - \cos 2\pi(x_o + c't)/L\right] + y_o, \tag{40}$$

where x_o and y_o are the coordinates of the trajectory at $t = 0$, and $c' = U - c$.

For the mid-point of successive streamlines method, if T is the time interval between maps, from the first map:

$$x_1 = UT + x_o,$$
$$y_1 = A_oL/2\pi U \left[\cos 2\pi x_o/L - \cos 2\pi (x_o + UT)/L\right] + y_o \tag{41}$$

and from the second map:

$$x_2 = UT + x_o,$$
$$y_2 = A_oL/2\pi U \left[\cos 2\pi (x_o - cT)/L - \cos 2\pi (x_o + c'T)/L\right] + y_o. \tag{42}$$

The coordinates of the trajectory at the time of the second map are, by this method:

$$x_m = UT + x_o,$$
$$y_m = A_oL/4\pi U \left[\cos 2\pi x_o/L - \cos 2\pi (x_o + UT)/L + \cos 2\pi (x_o - cT)/L - \cos 2\pi (x_o + c'T)/L\right] + y_o. \tag{43}$$

Applying the central tendency method, the coordinates of the trajectory at time T/2, as obtained from the first map, are:

$$x_1 = UT/2 + x_o$$
$$y_1 = A_oL/2\pi U \left\{\cos 2\pi x_o/L - \cos 2\pi[(UT/2) + x_o]/L\right\} + y_o. \quad (44)$$

The coordinates at the time of the secnnd map, obtained by moving the particle from its position at time T/2 to its position at time T along the streamlines of the second map are:

$$x_c = UT/2 + x_1,$$
$$y_c = A_oL/2\pi U \left\{\cos 2\pi (x_1 - cT)/L - \cos 2\pi [x_1 - cT + (UT/2)]/L\right\} + y_1. \quad (45)$$

or:

$$x_c = UT + x_o,$$
$$y_c = A_oL/2\pi U \left\{\cos 2\pi [x_o + (UT/2) - cT]/L - \cos 2\pi (x_o + c'T)/L + \cos 2\pi x_o/L - \cos 2\pi [(UT/2) + x_o]/L\right\} + y_o. \quad (46)$$

B. <u>Linearly varying amplitude</u> - If it is assumed that the amplitude factor varies linearly with time, it can be represented by:

$$A(t) = A_o + A_1 t, \quad (47)$$

where A_o is the value of the amplitude factor at time $t = 0$ and A_1 is its rate of change. From (39), it is seen that the equations describing the trajectory in this case are:

$$dx/dt = U, \qquad (48)$$
$$dy/dt = (A_o + A_1 t) \sin 2\pi (x - ct)/L$$

Solution of (48) yields for the true trajectory:

$$x = Ut + x_o,$$
$$y = L/2\pi c' [A_o \cos 2\pi x_o/L - (A_o + A_1 t) \cos 2\pi (c't + x_o)/L] + A_1 L^2/(2\pi c')^2 [\sin 2\pi (c't + x_o)/L - \sin 2\pi x_o/L] + y_o. \qquad (49)$$

In a manner analogous to that of the preceding section, results for the mid-point of successive streamlines method and for the central tendency method may be found.

It should be noted that in all the cases involving sinusoidal streamlines, the x-component of the wind was considered constant, so that both trajectory methods agree with the true value for the x-coordinate of the trajectory, the deviations occur only in the y-coordinate.

IV. Results

Table I illustrates the errors for a given set of numbers for the parameters describing the particular characteristic of the field of motion in question. It is seen from the illustration that the central tendency method produces the lesser error in all cases. It is concluded that to the extent to which the fields of motion noted in Table I are applicable to the atmosphere, the central tendency method produces smaller errors than the mid-point of successive streamlines.

TABLE I. Trajectory Errors After 12 Hours in Nautical Miles

Wind Field	Trajectory Computation Method: Mid-Point of Successive Streamlines	Central Tendency
A. Linear wind fields moving at 20 knots and coefficients $a = b = e = 0.1\ hr.^{-1}$		
Rotation	28	14
Deformation	40	20
Divergence	54	27
B. Stationary linear wind fields growing from 0 to $0.1\ hr.^{-1}$ in 12 hrs.		
Rotation	non-zero except at $x_o =$	0
Deformation	$y_o = 0$; error depends	0
Divergence	on starting position.	0
C. Sinusoidal wind field, wave length, 3770 n. miles; zonal wind, 60 knots; strongest N-S wind, 60 knots, c=20 knots.		
Start at trough or ridge	14	2
Start at point of inflection	34	5
D. Same as C., except that strongest N-S wind grows from 0 to 60 knots in 12 hours.		
Start at trough or ridge	114	11
Start at point of inflection	28	7

FIGURE 1

Maps A and B are twelve hours apart. The lines labelled h, h + 1, h + 2... are streamlines but may be considered to be height lines on a constant pressure surface. Particle starts at P at the time of Map A in the direction toward Q but ends at Q" in twelve hours moving along the dotted line in the lower chart. The distance PQ and PQ' are twelve hours of air motion.

FIGURE 2

Maps A and B are twelve hours apart. The lines labelled h, h + 1, h + 2.... are streamlines but may be considered to be height lines on a constant pressure surface. Particle starts at P at the time of Map A moving for six hours parallel to its streamlines to Q'. Position Q' is located on Map B and continued for six more hours parallel to streamlines on Map B to Q". The curve PQ' Q" is the twelve hour path.

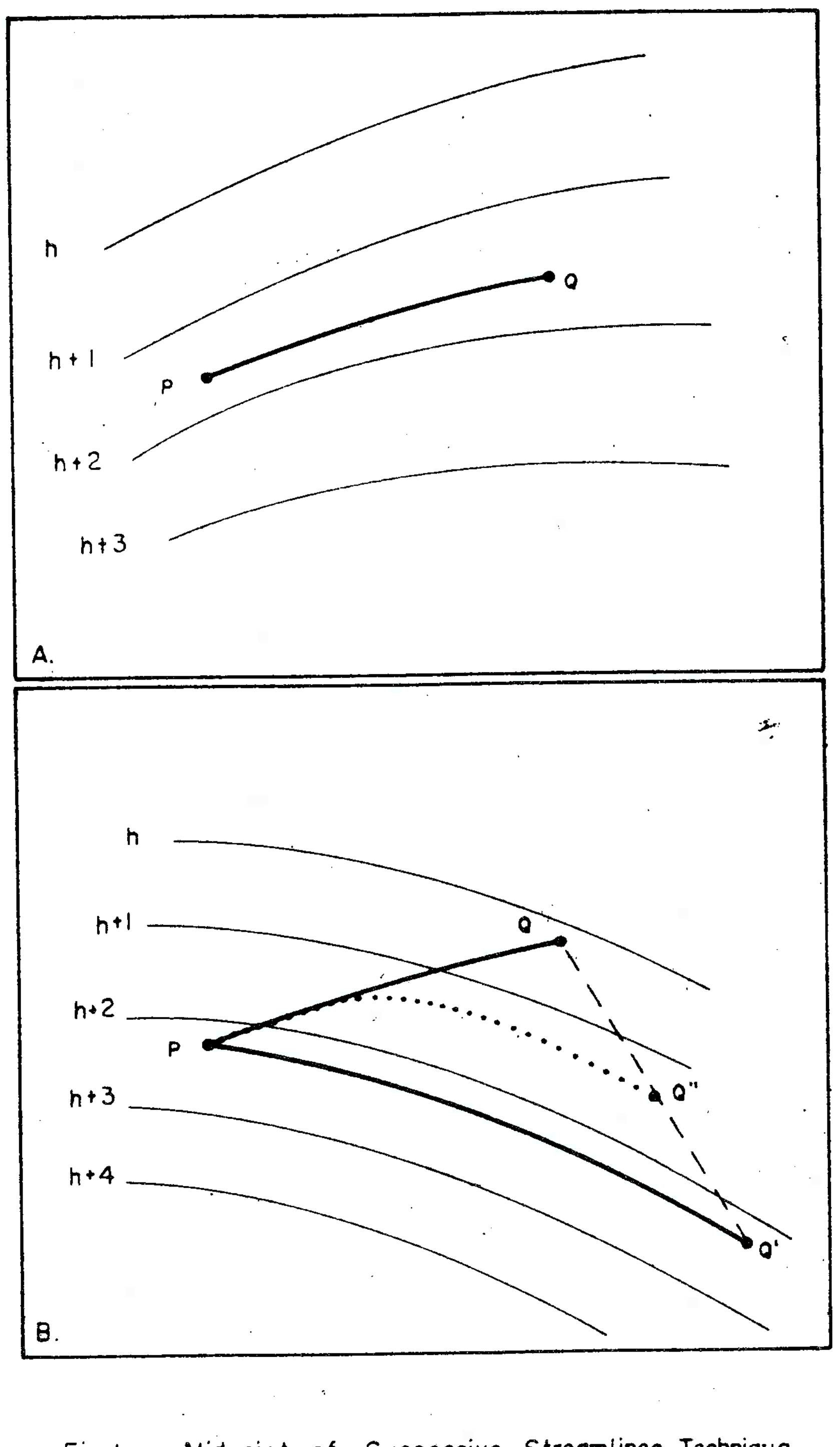

Fig. 1 Midpoint of Successive Streamlines Technique.

Method I

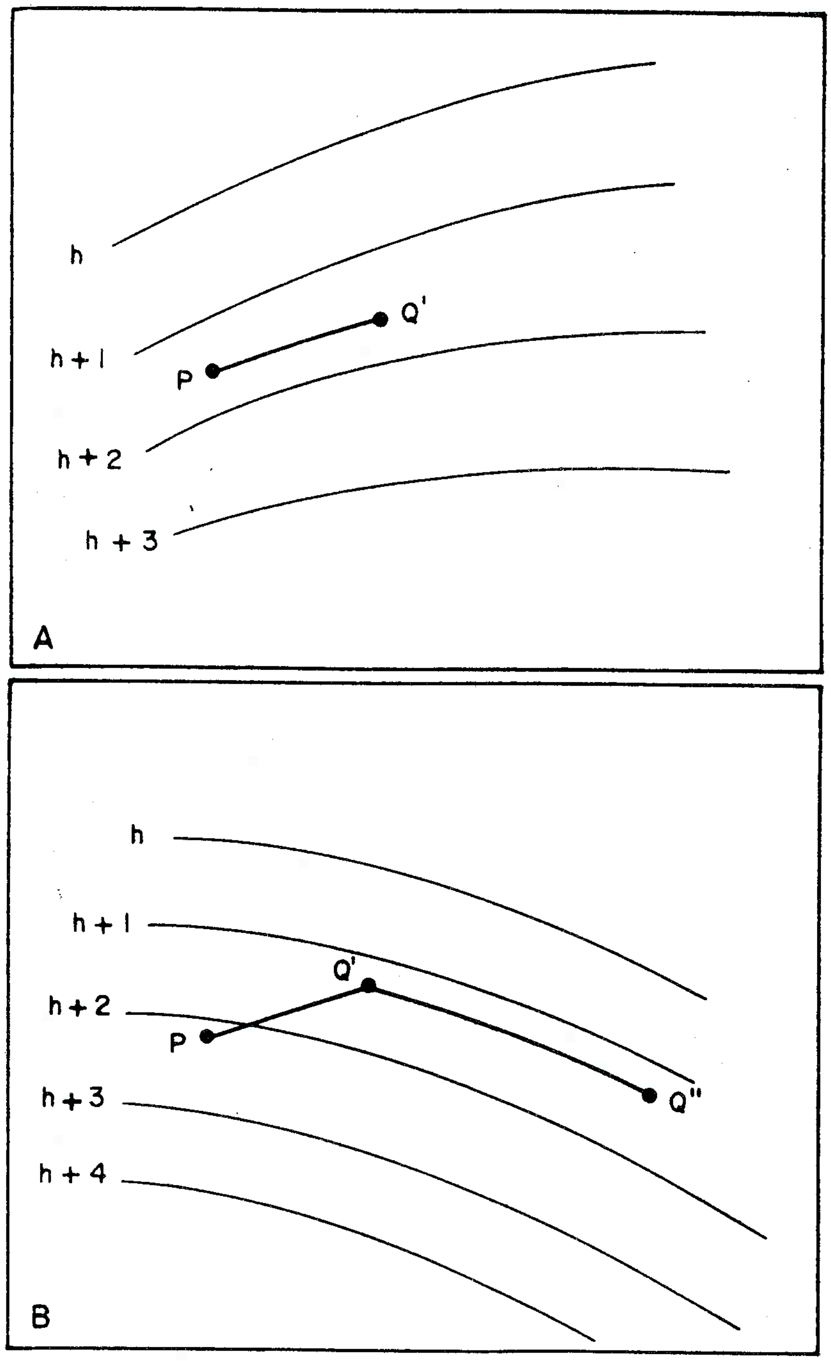

FIG. 2 CENTRAL TENDENCY METHOD